On the geometrical description of fractional Chern insulators based on static structure factor calculations

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We study the static structure factor of the fractional Chern insulator state. The averaged over Brillouin zone Fubini-Study (quantum distance) metric of the underlying non-interacting quantum system enters the quadratic form with small momenta in the expansion of the single particle part of the static structure factor. The form corresponds to the plasmonic part in the Laughlin case, and thus we find that the averaged over Brillouin zone Fubini-Study metric plays the role of "Landau level" metric in the framework of the geometrical description of fractional quantum Hall systems [F.D.M. Haldane, PRL 107, 116801 (2011)] and an effective (continuum) description of fractional Chern insulators. Assuming Laughlin-like correlations in the two body part of the static structure factor, we discuss the properties of the static structure factor in the long-distance limit, and analyze the conditions that have to be satisfied in order for the FQHE scenario [S.M. Girvin, A.H. MacDonald, and P.M. Platzman, Phys. Rev. B 32, 8458 (1985)] to occur. We discuss the relationship of the (averaged) Fubini-Study metric and Berry curvature, and illustrate their influence and correspondence in Haldane model based fractional Chern insulators.

I. INTRODUCTION

Chern insulators (CIs) [1] exhibit integer quantum Hall effect (IQHE) conductance quantization in the absence of the magnetic field due to non-trivial filled band structure with non-zero topological Chern number. Fractional Chern insulator (FCI) [2–4] is the name for CI with a partially filled band (which is akin to a Landau level) in the presence of strong interactions, which exhibit fractional quantum Hall effect (FQHE) conductance quantization.

What might be defined as an ultimate goal in the context of the FCI physics would be the understanding of the mechanism of creation of FCI states in order to be able to suggest the most convenient experimental settings, whether in cold atoms, crystal (solid-state) physics, or graphene structures for their realization. Interacting topological insulator [5–7] physics, which FCI is a time-reversal-symmetry-broken representative, is one of the major topics of the current research in the field of strongly correlated systems. FCI have been reported in many models both for fermionic systems [2–4, 8] and bosonic cold atomic models [9, 10]. Still there is no complete understanding why some model crystal systems are more convenient than others for particular FCI states. One way to resolve this problem would be to study the underlying quantum geometry of the crystal system and its influence on the gap function of FCI in the scope of the usual approximation in the FQHE physics: single mode approximation (SMA) [11].

In practice finding favorable conditions for FCI means achieving the understanding of the interplay of the statistics of the underlying particles, interactions, and the background lattice that produces the band with non-trivial topology characterized by a non-zero Chern number. We will focus our attention on the influence of the background lattice, and the structure of its particular band in which the FCI physics takes place. The band structure is characterized by two tensors that provide the description of the evolution of the quantum - mechanical state as it changes with the change of the Bloch momentum of the lattice: the Fubini-Study (FS) metric (or quantum distance) and the Berry curvature. The question of the role of the FS metric in the context of FCI was first raised in Ref. [12]. A recent work[13] showed how the FS metric is related and can be measured via the current noise spectrum. While there is a general understanding that an almost flat Berry curvature favors the emergence of a FCI phase (mimicking the constant magnetic field of FQHE), there is a little or no understanding what is the influence of the FS metric. We know that there is no quantization of its value when integrated over the Bloch momentum phase space - Brilllouin zone, like in the case of Berry curvature (that produces the Chern number). Thus quite generally its averaged value is dependent on the lattice parameters, and we cannot reduce the situation to the one of a single Landau level (FQHE) in which the relationship between, what we may identify as, Berry curvature and FS metric is fixed and not model dependent.

On the other hand we want to point out that there is the largely unexplored problem and question on the status of SMA for FCI. In the search for favorable conditions for FCI this is the most likely tool that is available if we want to understand the parameter (FS metric) dependence of the measure of the stability of the FCI state - its gap function. In this work we will pay special attention to the static structure factor (SSF), the norm of the approximate state, although "f(k)" (the oscillator strength) function, i.e. the expectation value of the energy of the system with respect to this state is also important and

needed. In the literature we find [14] a general statement on the nature of the expansion of the f(k) function for FCI; the leading term in the long distance expansion is quadratic in the low momentum due to the fluctuating Berry curvature in FCI. This necessarily means, in the scope of the SMA, that the SSF has the same leading behavior. We want to point out that the derivation in the same reference [14] was done with the usual averaging and long-wavelength limit procedures [15–17] in FCI, but which were not assumed or applied in the original work in Ref. [11] on the SMA in the context of FQHE. The behavior that was described in Ref. [11] is the leading behavior of the SSF and f(k) as quartic in small momenta, and this is, we may say, a hallmark of the FQHE behavior.

Thus it is natural to ask the following question: Whether, in a mean field picture at least we can talk about the scenario of Ref. [11] for FCI, or lower order terms in the SSF and f(k) that depend on the fluctuations in the Berry curvature and FS metric will determine the gap function. In this work we will study the SSF for FCI Laughlin-like states. In doing this one may resort to a numerical calculation of the SSF of FCI, which undoubtedly will shed most light on the nature of the expansion of the SSF - whether or not we have the quadratic term. Due to the approximations made and complicated expressions in Ref. [14] that needs to be checked for concrete FCI states. On the other hand, as will be described in this work, one may build a mean field picture (valid at least in the constant curvature and constant metric case) based on analytical considerations in the long-distance limit. By analyzing the single particle physics of the band with non-trivial Chern number and its consequences on the SSF of FCI, we can find out that the effective ("plasmonic") density-density force in the long-distance limit is inversely proportional to the averaged over Brillouin zone (BZ) (non-zero) diagonal element(s) of the FS metric. Thus the averaged over BZ FS metric plays the role of "Landau level" metric in the framework of the geometrical description of fractional quantum Hall systems [18] and an effective description of FCIs. In the effective description of the non-interacting CI band the density-density force will ensure that the long-distance fluctuations are suppressed - that the non-interacting system is gapped, and the density is uniform in the long-distance limit. By applying the Feenberg formula [19–21] interacting quantum liquid we can track down the influence of the FS metric on the ensuing coefficient of the quartic term of the SSF which will generalize the expression of Ref. [11] to the FCI case.

Thus the problem of calculating the SSF in the context of FCIs possesses many difficulties, and makes transparent the very nature of the FCIs' complex background i.e. fluctuating Berry curvature and fluctuating FS metric. The geometric picture of FQHE systems proposed in Ref. [18], in the context of FCIs may find the most non-trivial realization where also extrinsic part of the metric degree of freedom (not the one related to interactions) is non-trivial. In this work, a definite answer for the SSF behavior in the long-distance limit will not be given, but we will analyze the most relevant properties in the same limit by analytical means. It will be explained which conditions have to be fulfilled in order for the FQHE scenario (Ref. [11]) to occur in these systems. To illustrate our main conclusions we analyzed the underlying quantum geometry of the Haldane model [1] and the presence of bosonic FCI states in the phase diagram of the same model with on-site interaction (repulsion) only. We choose bosons in order to minimize the influence of the symmetry of the underlying lattice. The paper is organized as follows. In Section II the derivation of SSF in the context of FQHE is reviewed. In Section III the role of the FS metric in the context of FCIs and FCI SSF is identified. In the scope of a mean field approximation together with the assumption of the Laughlin-Jastrow correlations, the long-distance limit of the SSF for Abelian FCI states is calculated in Section IV. The possibility for the same scenario of Ref. [11] in the context of FCI states is discussed in the same section. In Section V the background degrees of freedom, the FS metric and Berry curvature, for the Haldane model based FCIs are analyzed. In the same section, Section V, the phase diagram of the interacting system of bosons (that live on the lattice defined by the Haldane model) is presented, and the identification of possible FCI regions and a comparison with the calculated background properties was made possible.

II. THE STATIC STRUCTURE FACTOR AND LAUGHLIN CASE

A. Static structure factor

We define static structure factor (SSF) as

$$s(q) = \frac{1}{V}\langle\Psi|\rho_{-q}\rho_q|\Psi\rangle, \tag{1}$$

where V is the volume of the system, $|\Psi\rangle$ is a normalized many-body wave function, and

$$\rho_q = \int \mathrm{d}\mathbf{r}\ \exp\{-\mathrm{i}\mathbf{q}\cdot\mathbf{r}\}\rho(\mathbf{r}), \tag{2}$$

where

$$\rho(\mathbf{r}) = \sum_{i=1}^{N}\delta^{(2)}(\mathbf{r}-\mathbf{r}_i), \tag{3}$$

is the density operator of the system of N particles with coordinates $\{\mathbf{r}_i; i=1,\ldots,N\}$. If we introduce the radial distribution function,

$$g(|\mathbf{r}_1-\mathbf{r}_2|) = \frac{N(N-1)}{n^2}\int d^2\mathbf{r}_3\cdots\int \mathrm{d}\mathbf{r}_N|\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_N)|^2, \tag{4}$$

where $n = N/V$ is the averaged density, we can rewrite Eq. 1 as

$$s(q) = n + n^2 \int \mathrm{d}\mathbf{r}\; g(r) \exp\{-\mathrm{i}\mathbf{q}\mathbf{r}\} \tag{5}$$

We will call the first term in $s(q)$ single particle part and the second one two-body (correlation) part.

In the second quantization language with creation and annihilation operators in the coordinate space, $\hat{\Psi}^\dagger(\mathbf{r})$ and $\hat{\Psi}(\mathbf{r})$,

$$[\hat{\Psi}(\mathbf{r}), \hat{\Psi}^\dagger(\mathbf{r}')]_\pm = \delta^{(2)}(\mathbf{r} - \mathbf{r}'), \tag{6}$$

where + sign denotes a commutator in the case of bosons, and − sign denotes an anticommutator in the case of fermions, we can easily work out the corresponding expression for $s(q)$:

$$\begin{aligned} s(q) &= \frac{1}{V}\langle\Psi| \int \mathrm{d}\mathbf{r}\; \hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r})\; |\Psi\rangle \\ &+ \frac{1}{V}\int \mathrm{d}\mathbf{r} \int \mathrm{d}\mathbf{r}'\; \langle\Psi|\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}^\dagger(\mathbf{r}')\hat{\Psi}(\mathbf{r}')\hat{\Psi}(\mathbf{r})|\Psi\rangle \times \\ &\exp\{-\mathrm{i}\mathbf{q}(\mathbf{r} - \mathbf{r}')\}. \end{aligned} \tag{7}$$

In other words, by using the basic algebra of particle operators i.e. (anti)commuting relation in Eq. 6 we can recover separate single part (first term of Eq. 7) and the two-body part (second term of Eq. 7).

B. Quantum Hall case - Laughlin case

We would like to calculate the same quantity SSF, defined in Eq. 1 where $|\Psi\rangle$ is the normalized Laughlin wave function. With respect to Ref. [22], we use the definition of SSF given by Eq. 1, in which we divide the density-density correlator by volume (V) instead of the number of particles (N). The definition used in Ref. [22] corresponds to the norm of the model state used in SMA. In the following we will summarize the results of weak coupling plasma approach as described in Refs. [21] and [23].

If we assume weak coupling in the Laughlin plasma approach [24, 25] we can study expansions of expectation values as in Eq. 1 in terms of the 2D Coulomb plasma interaction,

$$v(q) = -\frac{4\pi m}{|q|^2}, \tag{8}$$

where positive integer m is connected to the filling factor ν of the quantum Hall system as $\nu = 1/m$. Because of the expected screening of the 2D plasma the contributions in the weak coupling perturbative approach can be organized as in

$$s(q) = \frac{s_0(q)}{1 - v(q)s_0(q)}, \tag{9}$$

where $s_0(k)$ represents the contribution from "irreducible diagrams". In the lowest order in m, $s_0(q) \approx n = 1/2\pi m$. We set the magnetic length to one, $l_B \equiv 1$. The single particle contribution, n is contained in Eq. 9.

In the lowest Landau level (LLL) we can define the projected density,

$$\tilde{\rho}_q = \int \mathrm{d}^2 z \Psi^\dagger(z) \exp\{\mathrm{i}q\frac{\partial}{\partial z}\} \exp\{\mathrm{i}\frac{q^* z}{2}\}\Psi(z), \tag{10}$$

where $\Psi(z)$ are second quantized operators in the LLL,

$$\Psi = \sum_{l=0}^{\infty} \hat{a}_l \frac{1}{\sqrt{2\pi 2^l l!}} z^l \exp\{-\frac{1}{4}|z|^2\}, \tag{11}$$

$z = x + \mathrm{i}y$ is the complex 2D coordinate and $[\hat{a}_l, \hat{a}^\dagger_m]_\pm = \delta_{l,m}$. The derivatives in Eq. 10 act only on the holomorphic part (dependent only on z) of $\Psi(z)$. We have

$$\begin{aligned} &\tilde{\rho}_{-q}\tilde{\rho}_q = \\ &\int \mathrm{d}^2 z \int \mathrm{d}^2 z' [\exp\{-\mathrm{i}q^* \frac{\partial}{\partial z'^*}\} \exp\{-\mathrm{i}\frac{q z'^*}{2}\}\Psi^\dagger(z')]\Psi(z') \\ &\times \Psi^\dagger(z) \exp\{\mathrm{i}q\frac{\partial}{\partial z}\} \exp\{\mathrm{i}\frac{q^* z}{2}\}\Psi(z), \end{aligned} \tag{12}$$

Similarly to the unprojected case, Eqs. 6 and 7, we can use the (anti)commutation relations of projected single particle operators, $\Psi(z)$, to get the division in the single and two-body part of SSF. The single particle part of the SSF corresponds to the diagonal contribution either in momentum or coordinate space that refers to a single particle. We will denote by $\langle\tilde{\rho}_{-q}\tilde{\rho}_q\rangle|_{\mathrm{single}}$, the single part of the projected SSF. Due to

$$[\Psi(z'), \Psi^\dagger(z)]_\pm = \frac{1}{2\pi} \exp\{\frac{z^* z'}{2}\} \exp\{-\frac{|z|^2}{4}\} \exp\{-\frac{|z'|^2}{4}\}, \tag{13}$$

i.e. equality of the (anti)commutator to the LLL delta function, the single particle part of the projected SSF is

$$\langle\tilde{\rho}_{-q}\tilde{\rho}_q\rangle|_{\mathrm{single}} = n \exp\{-\frac{|q|^2}{2}\}. \tag{14}$$

The two-particle correlations (i.e. when $z' \neq z$) stay the same (examine Eqs. 12 and 7 with Laughlin wave function in that case) and therefore the expression for the projected SSF is

$$\tilde{s}(q) = n \exp\{-\frac{|q|^2}{2}\} + \frac{s_0(q)}{1 - v(q)s_0(q)} - n. \tag{15}$$

The important correction to the $s_0(q) \approx n$ approximation is made of a bubble diagram in the next order in m; the interaction lines (two of them) are screened. Therefore the contribution is

$$\delta_q = n^2 \frac{1}{2} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} V_{\mathrm{eff}}(\mathbf{q} - \mathbf{k}) V_{\mathrm{eff}}(\mathbf{k}), \tag{16}$$

where $1/2$ is a symmetry factor and

$$V_{\text{eff}}(\mathbf{q}) = \frac{V(\mathbf{q})}{1 - nV(\mathbf{q})} = -\frac{4\pi m}{|\mathbf{q}|^2 + 2}. \tag{17}$$

The correction for $\mathbf{q} = \mathbf{0}$ is

$$\delta_0 = n\frac{m}{2}. \tag{18}$$

We checked that the $|\mathbf{q}|^2$ correction to δ_q in $s_0(q) \approx n + \delta_q$ will not change the resulting behavior due to the inclusion of $s_0(q) \approx n + \delta_0$ in Eq. 15:

$$\tilde{s}(q) \approx \frac{(m-1)}{8}|q|^4 n. \tag{19}$$

If we assume that the analytical continuation for $m \geq 1$ is valid we recovered the formula of Ref. [22] that follows from the 2D plasma compressibility rule.

III. FCI CASE - SINGLE PARTICLE CONTRIBUTION

We start with a lattice system in which Bloch Hamiltonian $H(p)$ is diagonal with eigenvalues, $\epsilon_m(p)$, where $m = 1, \ldots, r$ denotes the band index, and eigenstates, $u^m_{\alpha,p}$, where $\alpha = 1, \ldots, r$ denotes the orbital index in a unite cell, and p is the Bloch momentum, i.e.

$$\sum_\beta h_{\alpha,\beta}(p) u^m_{\beta,p} = \epsilon_m(p) u^m_{\alpha,p}. \tag{20}$$

In order to study the partially filled band with Chern number equal to one in which a FCI state occurs, we confine our description to that band, and drop the index m in the following. In the case of FCI we take the projected density to a single band to be defined as in Ref. [17] i.e.

$$\tilde{\rho}_q = \sum_p u^*_{\alpha,p} u_{\alpha,p+q} \gamma^\dagger_p \gamma_{p+q}, \tag{21}$$

where the summation on the repeated orbital Greek index (α) is assumed, p and q are Bloch momenta, and γ_p are normal mode operators.

Therefore

$$\begin{aligned}&\tilde{\rho}_{-q}\tilde{\rho}_q = \\ &\sum_{p_1,p_2} u^*_{\alpha,p_1} u_{\alpha,p_1-q} u^*_{\alpha,p_2} u_{\alpha,p_2+q} \times \\ &\gamma^\dagger_{p_1}\gamma_{p_1-q}\gamma^\dagger_{p_2}\gamma_{p_2+q}\end{aligned} \tag{22}$$

and because

$$[\gamma_{p_1-q}, \gamma^\dagger_{p_2}]_\pm = \delta_{p_1-q,p_2} \tag{23}$$

we have

$$\begin{aligned}&\langle\tilde{\rho}_{-q}\tilde{\rho}_q\rangle|_{\text{single}} = \\ &\sum_p u^*_{\alpha_1,p} u_{\alpha_1,p-q} u^*_{\alpha_2,p-q} u_{\alpha_2,p}\, n_p,\end{aligned} \tag{24}$$

where $n_p \equiv \langle\gamma^\dagger_p\gamma_p\rangle$ is the occupation of the Bloch momentum p in the many-body FCI state.

After a few steps, which are described in Appendix A, the expansion in small momentum q of the single part of the unprojected SSF for the FCI state is

$$\langle\tilde{\rho}_{-q}\tilde{\rho}_q\rangle|_{\text{single}} = n - q_i q_j n\overline{g^{FS}_{ij}} + \frac{q_i q_j q_k}{2} n\overline{\partial_k g^{FS}_{ij}} + o(q^4) \tag{25}$$

where we assumed summations over repeated indices,

$$\begin{aligned}&g^{FS}_{ij}(p) = \frac{1}{2}[\partial_i u_{\alpha_1,p}\partial_j u^*_{\alpha_1,p} + \partial_j u_{\alpha_1,p}\partial_i u^*_{\alpha_1,p} - \\ &\partial_i u_{\alpha_1,p} u^*_{\alpha_1,p} u_{\alpha_2,p}\partial_j u^*_{\alpha_2,p} - \partial_j u_{\alpha_1,p} u^*_{\alpha_1,p} u_{\alpha_2,p}\partial_i u^*_{\alpha_2,p}]\end{aligned} \tag{26}$$

is the Fubini-Study metric, see, for example, Ref. [12] for the definition, and

$$\overline{g^{FS}_{ij}} = \frac{\sum_p g^{FS}_{ij}(p) n_p}{n} \tag{27}$$

and

$$\overline{\partial_k g^{FS}_{ij}} = \frac{\sum_p \partial_k g^{FS}_{ij}(p) n_p}{n} \tag{28}$$

are averages over the whole BZ. If we assume $n_p =$ constant, which we expect to hold at least approximately in the FCI state, see Section IV B (Eq. 48) for an explanation of this point, we can substitute the averages over occupation number to the ones over Brillouin zone. Because of the periodicity in the k-space we have that the averaged derivatives over the metric (Eq. 28) are equal to zero in the case of FCIs. Also we can argue that, assuming the inversion symmetry, the expression in Eq. 28 is zero. Thus

$$\langle\tilde{\rho}_{-q}\tilde{\rho}_q\rangle|_{\text{single}} = n - q_i q_j n\overline{g^{FS}_{ij}} + o(q^4). \tag{29}$$

Comparing with the Laughlin case, Eqs. 14 and 15, we expect that the quadratic term corresponds to the plasmonic part that has to be canceled in the projected (to the band) SSF if the FQHE scenario occurs in the context of FCI.

Continuing the analysis of the single part to the quartic order we have

$$\begin{aligned}\langle\tilde{\rho}_{-q}\tilde{\rho}_q\rangle|_{\text{single}} = \quad & n - q_i q_j n\ \overline{g^{FS}_{ij}} + q_i q_j q_k q_l n\ \overline{h_{ijkl}} \\ & + \text{higher order terms}\end{aligned} \tag{30}$$

where

$$\begin{aligned}h_{ijkl} \ = \ & \frac{1}{4!}[u_{\alpha,p}\partial_k\partial_i\partial_j\partial_l u^*_{\alpha,p} + \partial_k\partial_i\partial_j\partial_l u_{\alpha,p}\ u^*_{\alpha,p} \\ & +3\partial_i u_{\alpha_1,p} u^*_{\alpha_1,p}\ u_{\alpha_2,p}\partial_k\partial_j\partial_l u^*_{\alpha_2,p} + \\ & +3\ \partial_i\partial_k\partial_l u_{\alpha_1,p} u^*_{\alpha_1,p}\ u_{\alpha_2,p}\partial_j u^*_{\alpha_2,p} + \\ & +6\ \partial_i\partial_k u_{\alpha_1,p} u^*_{\alpha_1,p}\ u_{\alpha_2,p}\partial_j\partial_l u^*_{\alpha_2,p}],\end{aligned} \tag{31}$$

and

$$\overline{h_{ijkl}} = \frac{\sum_p h_{ijkl}(p) n_p}{n}. \tag{32}$$

In the analogy with the Laughlin case we would expect that the coefficient of the quartic term is a "square of metric", but the tensor h_{ijkl} can not be greatly simplified without further assumptions. Assuming n_p = constant, we can shift the sum over p to the one over $P = p - q/2$, and discuss the product, $u^*_{\alpha,P+q/2}u_{\alpha,P-q/2}$ separately. In this case the quartic coefficient becomes,

$$\overline{h^{FCI}_{ijkl}} = \frac{\overline{g^{FS}_{ij}g^{FS}_{ij}}}{4} + \frac{\overline{g_{ijkl}}}{4}, \tag{33}$$

where g_{ijkl} is a tensor. By analyzing $u^*_{\alpha,P+q/2}u_{\alpha,P-q/2}$ product and rewriting it as

$$u^*_{\alpha,P+\frac{q}{2}}u_{\alpha,P-\frac{q}{2}} = \exp\{-\mathrm{i}q_iA_i\}f(u,u^*), \tag{34}$$

where $A_i = -iu^*_{\alpha,p}\partial u_{\alpha,p}$, the Berry connection, we can find out the expression for the g_{ijkl} tensor in its gauge invariant form,

$$\begin{aligned} 2g_{ijkl} \;=\; & -\frac{1}{3}(\partial_iu_{\alpha,p}\partial_j\partial_k\partial_lu^*_{\alpha,p} + \partial_i\partial_j\partial_ku_{\alpha,p}\;\partial_lu^*_{\alpha,p}) \\ & +\frac{1}{3}iA_i(u_{\alpha,p}\partial_j\partial_k\partial_lu^*_{\alpha,p} - \partial_i\partial_k\partial_lu_{\alpha,p}u^*_{\alpha,p} \\ & +\partial_i\partial_ku_{\alpha,p}\partial_ju^*_{\alpha,p} - \partial_iu_{\alpha,p}\partial_j\partial_lu^*_{\alpha,p}) \\ & +4A_iA_j\partial_ku_{\alpha,p}\partial_lu^*_{\alpha,p} - 2A_iA_jA_kA_l. \end{aligned} \tag{35}$$

Even the condition on the constancy of the metric, $\partial_i\partial_jg^{FS}_{kl} = 0$, does not simplify the form of the g_{ijkl} tensor and the coefficient h^{FCI}_{ijkl}. If we choose the LLL basis of Ref. 26, for which $g^{FS}_{11} = g^{FS}_{22} = \overline{B}/2$, where $\overline{B}$ is the averaged Berry curvature, i.e.

$$\overline{B} = \frac{\sum_p B}{A_{\mathrm{BZ}}} = \frac{2\pi C}{A_{\mathrm{BZ}}}, \tag{36}$$

where A_{BZ} is the area of the BZ, and $g^{FS}_{12} = g^{FS}_{21} = 0$, we expect to recover the "square of metric" form of the coefficient, otherwise, for a general FCI, we expect Eq. 33 to hold.

IV. SSF FOR FCI

A. A mean field picture

In the following we will consider a possibility that the absence of the quadratic term in the expansion of the projected SSF in the FQHE also occurs in the context of the FCI physics. In a mean field picture we may expect that the FCI system *in the long-wavelength limit* is a system with density $n = 1/2\pi ml^2_B$, fixed by the value of $l^2_B \equiv \overline{B}$ [15] (i.e. averaged Berry curvature i.e. Chern number), and that two body correlations are described in the same limit with an effective long-range density-density force,

$$v^{FCI}_{MF}(q) = -\frac{2\pi m}{q_iq_j\overline{g^{FS}_{ij}}}, \tag{37}$$

in order for the cancelation to occur.

In the following we will consider that

$$\begin{aligned} &\overline{g^{FS}_{11}} = \overline{g^{FS}_{22}} = g \geq \frac{\overline{B}}{2} \text{ and} \\ &\overline{g^{FS}_{12}} = \overline{g^{FS}_{21}} = 0, \end{aligned} \tag{38}$$

i.e. the averaged over BZ FS metric is diagonal with diagonal element equal to g. The value of g has the lower bound, $\bar{B}/2$, as explained in Ref. [12]. The diagonal form of the averaged metric will hold in the context of FCI states based on the Haldane model Chern insulator, when the assumption n_p = constant is applied.

Thus in calculating the SSF for FCI, in the approximation we adopted, either we can assume that the system in the long-wavelength limit is described by the Laughlin wave function (or its generalizations with in general "incongruent" relationship between flux and particle positions or unknown short distance behavior), and considering only the long-wavelength domain apply the plasma approach of Section II, or equivalently we can assume that we have a bosonic quantum liquid system with the long-range force to which we can apply the Feenberg formula [19, 20] (Eq. 9) in this limit.

Either way, assuming the analyticity in m and repeating the steps of Eqs. 16-19, we find that the two particle part of the SSF of FCI in this limit behaves as

$$\begin{aligned} &\lim_{q\to 0}\frac{s^{FCI}_0(q)}{1 - v^{FCI}_{MF}(q)s^{FCI}_0(q)} \to \\ &g\;n|q|^2 + g^2\;n|q|^4(\frac{m}{4}\frac{\overline{B}}{g} - 1). \end{aligned} \tag{39}$$

Within the assumptions made (and that we work with the FS metric, $\overline{g^{FS}_{11}} = \overline{g^{FS}_{22}} = g$) we can conclude that the form of the projected to a band SSF for a FCI state is

$$\begin{aligned} \tilde{s}^{FCI}(q) = \quad & \frac{g^2\;n|q|^4}{2}(m\frac{\overline{B}}{2g} - 2) \\ & +n\;q_iq_jq_kq_l\frac{\overline{g^{FS}_{ij}g^{FS}_{kl}} + \overline{g_{ijkl}}}{4}. \end{aligned} \tag{40}$$

In the case $g = \overline{B}/2$ we expect to recover the usual QHE form.

B. Discussion

According to the Ref. [14] fluctuating Berry curvature will produce the quadratic term in the expansion of the f_k function (Ref. [22]) in the SMA for the FCI. The authors concluded that necessarily the projected SSF of the FCI has to have the leading quadratic term in order to have a finite gap in the SMA.

We applied a mean field approach in calculating two-body correlations for the SSF of FCI. It is likely that only an exact numerical calculation based on a concrete FCI

state may determine whether the quadratic term in the expansion of the SSF is present or that the quartic term is dominant in determining the physics and energetics of FCIs (as in the usual FQHE case).

If we nevertheless maintain that in a mean field picture the formula (40) enters the expression for the gap function (as a dominator - a norm of the SMA state) in the SMA of the FCI state, we can conclude that large $g \gg \overline{B}/2$ may induce an instability ($\tilde{s}^{FCI}(q)$ will become negative, which cannot be true for a positive definite quantity) towards a gapless state ($\tilde{s}^{FCI}(q) \sim q^3$). In reality we might expect that either large discrepancy between g and the lower bound $\overline{B}/2$, or strong fluctuations of the FCI metric may lead to a gapless state. In order to investigate this question we calculated $\overline{g_{ij}^{FS}}$, and the standard deviation of the FS metric from its averaged value, $\overline{g_{ij}^{FS}}$, in the Brillouin zone for a particular model, and we will present this in Section V.

To understand better (see also [27–29]) the absence of the quadratic term in the FQHE we will discuss the case $g = \overline{B}/2$ and B(Berry curvature) $= \bar{B}$ in the FCI context. Expanding the expression in Eq. 21 for the projected density to a single band we have for the linear term in q:

$$\tilde{\rho}_q|_{\text{linear}} = q^k \sum_p \{\mathrm{i}A_k(p)\gamma_p^\dagger\gamma_p + \gamma_p^\dagger \frac{\partial}{\partial p_k}\gamma_p\} \equiv q^k T_k. \quad (41)$$

In the first quantization picture the operator T_k is

$$T_k = \sum_{i=1}^{N} \{\mathrm{i}A_k(p_i) + \frac{\partial}{\partial p_i^k}\}, \quad k = x, y \quad (42)$$

Using the complex representation we can rewrite the linear term in Eq. 41 in the radial gauge as

$$\sum_{i=1}^{N} \{q(\frac{\bar{B}}{4}p_i^* + \frac{\partial}{\partial p_i}) + q^*(-\frac{\bar{B}}{4}p_i + \frac{\partial}{\partial p_i^*})\}. \quad (43)$$

The solution must be of the form,

$$\Psi_0 = f(\{p_i\}) \exp\{-\frac{1}{4}\sum |p_i|^2\}, \quad (44)$$

i.e. belong to the LLL, and the operators, $R_i = (\overline{B}/4)p_i^* + \partial/\partial p_i$ and $R_i^\dagger = (\overline{B}/4)p_i - \partial/\partial p_i^*$, we recognize, corresponding to guiding center coordinates in the momentum representation of the QH problem. They make the simple bosonic algebra, $[a_i, a_i^\dagger] = 1$, if we take $\bar{B} = 1$ and $a_i = \sqrt{2}R_i$ and $a_i^\dagger = \sqrt{2}R_i^\dagger$ of the LLL for each particle. In this representation SSF can be expressed as

$$\sum_{i,j} \langle : \exp\{q^* R_i^\dagger - qR_i\} :: \exp\{-q^* R_j^\dagger + qR_j\} : \rangle - \sum_i \langle : \exp\{q^* R_i^\dagger - qR_i\} : \rangle \sum_j \langle : \exp\{-q^* R_j^\dagger + qR_j\} : \rangle, \quad (45)$$

where : : sign denotes the normal ordering. This definition implies the usage of the density operators in the SSF calculation that differs by the factor $\exp\{-|q|^2/2\}$ from the usual [22] operators. Nevertheless, this trivial difference should not affect the absence of the quadratic term, and when applying the expression in Eq. 45, we see that, after a ground state value substraction i.e. normal ordering - see Appendix for explanation of this point, the quadratic term will not exist if and only if

$$\sum_i R_i \Psi_0 = 0. \quad (46)$$

We know that this is satisfied in the disk and spherical geometry of a continuum system [28, 30], and expect to hold even in the lattice system, because the generator of translation should annihilate the ground state which is a homogenous, liquid state.

Therefore it is the existence of a homogenous ground state that is annihilated by the translation generator plus the existence of the bosonic algebra of the LLL that ensures the absence of the quadratic term, and appearance of the leading quartic term in the FQHE.

If we believe that the same scenario will happen in FCI we may consider the possibility that locally, in the BZ, even for varying curvature, we can have the bosonic algebra;

$$R_i = \frac{\bar{B} + \delta B}{4}p_i^* + \frac{\partial}{\partial p_i} \text{ and } R_i^\dagger = \frac{\bar{B} + \delta B}{4}p_i - \frac{\partial}{\partial p_i^*}, \quad (47)$$

where δB is a weakly dependent function on $\mathbf{p}$. With the condition that the FCI state must satisfy,

$$\sum_i R_i \Psi_{\text{FCI}}(\{\mathbf{p}_i\}) = 0, \quad (48)$$

and the normal ordering prescription for the bosonic algebra at each $\mathbf{p}$, the quadratic term will be absent in the low-momentum SSF expansion.

As we will see in a particular example of a bosonic FCI state based on the Haldane model at and around the point, $\mathbf{p} = (0, 0)$ in the BZ, which is the long-distance expansion point, in this particular model, the Berry curvature is zero. Thus the effective form of the ground state in this long-wavelength limit (when the particle momenta are small) may be of the Jastrow -Laughlin form, i.e.

$$\prod_{i<j} |p_i - p_j|^\gamma, \quad (49)$$

where γ is a constant, in order to satisfy Eq. 48 in the limit for which the Berry curvature and metric matrix elements, see Figs. 5 and 6 below, are zero. With the assumption that the cancelation of the quadratic terms in the small momentum expansion of the projected SSF occurs we expect $\gamma = m\overline{B}/2g$.

V. FCIS BASED ON THE HALDANE MODEL

A. Exact diagonalization results

The Haldane honeycomb model [1] is the first studied example of a Chern insulator. Several numerical evidences of a robust FCI have been reported for bosons with on-site repulsion on such a lattice [9, 31, 32]. Weaker FCIs have also been observed when bosons are replaced by fermions [33] (Note that the on-site interaction is then replaced by a nearest neighbor interaction). We will use the honeycomb lattice layout of Ref. [2], as shown in Fig. 1. The one-body Hamiltonian can be written in Bloch form as $h(\mathbf{k}) = d_0\mathbb{I} + \sum_i d_i\sigma_i$ using the Pauli matrices and where

$$\begin{aligned} d_0 &= 2t_2\cos\phi\ (\cos k_x + \cos k_y + \cos(k_x+k_y))\,, \\ d_x &= t_1\,(1+\cos(k_x+k_y)+\cos k_y)\,, \\ d_y &= -t_1\,(\sin(k_x+k_y)+\sin k_y)\,, \\ d_z &= M\ +\ 2t_2\sin\phi\ (\sin k_x+\sin k_y-\sin(k_x+k_y))\,. \end{aligned} \tag{50}$$

M (resp. $-M$) is the chemical potential added to the A (resp. B) sites, t_1 is the amplitude of the (real) nearest neighbor hopping term and $t_2\exp(\mathrm{i}\phi)$ the complex amplitude if the next nearest neighbor hopping term. The two components of the lattice momentum $\mathbf{k}$ are defined as $k_x = \mathbf{k}\cdot\mathbf{e}_1$ and $k_y = \mathbf{k}\cdot\mathbf{e}_2$, where $\mathbf{e_1}$ and $\mathbf{e_2}$ are the lattice vectors. For our numerical calculations, we set $t_1 = t_2$. The Haldane model has two bands. If $M/t_1 > 3\sqrt{3}\sin(\phi)$, the two bands are trivial. If $M/t_1 < 3\sqrt{3}\sin(\phi)$ then each band carries a non-zero Chern number (either $C=+1$ or $C=-1$).

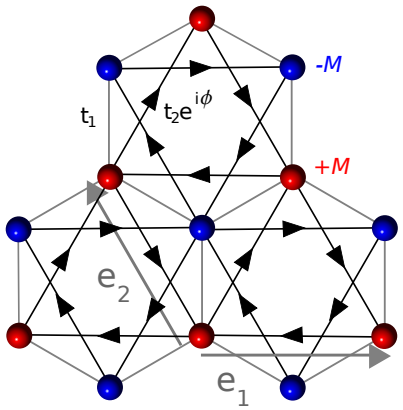

FIG. 1: The Haldane model on the honeycomb lattice with A (in red) and B (in blue) sublattices. The lattice translation vectors are $\mathbf{e}_1$ and $\mathbf{e}_2$. The amplitude of the nearest neighbor hopping is t_1 and the next nearest neighbor hopping is $t_2\exp(\mathrm{i}\phi)$ (in the direction of arrows). The sublattice chemical potential is set to $+M$ on A sites and $-M$ on B sites.

We consider N bosons on the Haldane honeycomb model with a lattice of N_x unit cells in the $\mathbf{e}_1$ direction and N_y unit cells in the $\mathbf{e}_2$ direction. The filling factor is thus defined as $\nu = N/(N_x\cdot N_y)$. We add on-site Hubbard-type density-density interaction $H_{\rm int} = \sum_i :n_in_i:$, where the sum runs over all the sites. To focus on the band topological properties, we use the flat-band approximation described in Ref. [4]: We start from the original Bloch Hamiltonian $h(\mathbf{k}) = \sum_{\alpha=1}^{2} E_\alpha(\mathbf{k})\mathcal{P}_\alpha(\mathbf{k})$ where $E_\alpha(\mathbf{k})$ and $\mathcal{P}_\alpha(\mathbf{k})$ are the dispersion and the projector onto the α-th band, respectively. Then we focus on the lowest band and consider the effective one-body flat band Hamiltonian $h_{\rm eff}(\mathbf{k}) = \mathcal{P}_1(\mathbf{k})$. From a physical perspective, it means that we set the band gap to infinity and we make the lowest band completely flat. In this approximation, the effective many-body hamiltonian writes $H_{\rm eff} = \mathcal{P}_1 H_{\rm int}\mathcal{P}_1$.

To study the stability of the FCI phase, we focus on the energy spectrum. In the FCI regime at filling factor $\nu = 1/2$, the Laughlin-like state on a torus geometry is characterized by two almost degenerate low energy states separated by a large from higher energy excitations. A typical low energy spectrum is shown in the left panel of Fig. 2). The energy splitting between the two lowest energy states is called the spread δ. In the case of FQHE, the spread should be equal to zero due to the center of mass degeneracy. A necessary condition to be able to distinguish the two lowest energy states is δ to be smaller than the gap Δ (defined as the energy difference between the third and the second lowest energy levels). Another necessary condition to claim a Laughlin-like is hosted in this system is related to the quantum number of the two lowest energy states: If they are associated to a Laughlin-like state, they should be given by the counting principle described in Ref. [17].

We have computed the phase diagram when tuning ϕ and M at filling $\nu = 1/2$ for two different system sizes: $N=8$ on a $N_x = N_y = 4$ lattice (Fig. 2) and $N=10$ on a $N_x = 5, N_y = 4$ lattice (Fig. 3). We show both the gap Δ and the spread δ (actually $1-\min(\delta,\Delta,1)$ such that we have 1 when the spread is 0 and 0 if $\delta > \Delta$). When the band structure parameters are set to values leading to a trivial band, we clearly see that the FCI phase completely disappears. In the non-trivial region, both system sizes suggest a robust Laughlin-like state around $M=0$ and $\phi = 0.11 - 0.12\pi$.

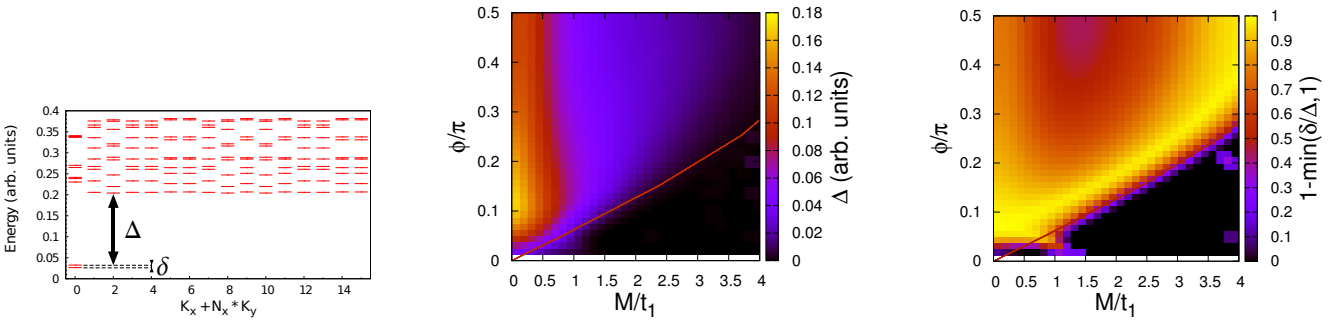

FIG. 2: *Left panel:* Typical low energy spectrum for the FCI Haldane model as a function of the linearized two-dimensional momentum (k_x, k_y). Here we have set $N = 8$, $N_x = N_y = 4$, $M = 0$ and $\phi = 0.12\pi$. The spread δ is the splitting between the two lowest energy states, corresponding to the twofold degenerate Laughlin states. The gap Δ is defined as the energy difference between the third lowest energy and the second lowest energy, irrespective of the momentum sector. *Middle panel:* The energy gap Δ as a function of the two tight-binding model parameters ϕ and M/t_1, for $N = 8$ bosons on a $N_x = 4, N_y = 4$. The gap is set to zero when the two lowest energy states are not in the expected momentum sectors of the Laughlin state, here at $(K_x = 0, K_y = 0)$ for both states. The red line denotes the separation between the Chern insulator phase (upper part) and the trivial phase (lower part). *Right panel:* The corresponding spread δ (displayed as $1 - \min(\delta/\Delta, 1)$) as a function of ϕ and M/t_1. A color value of 1 would correspond to a perfectly twofold degenerate groundstate (i.e. $\delta = 0$).

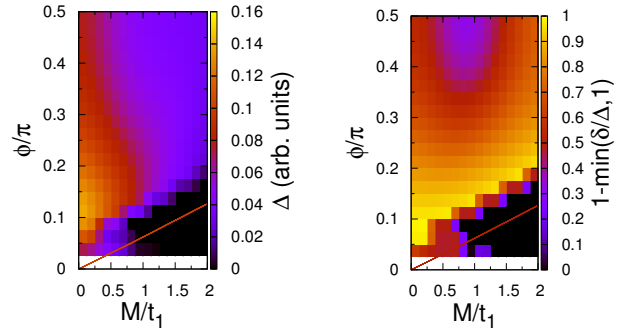

FIG. 3: *Left panel:* The energy gap Δ as a function of the two tight-binding model parameters ϕ and M/t_1, for $N = 10$ bosons on a $N_x = 5, N_y = 4$. The gap is set to zero when the two lowest energy states are not in the expected momentum sectors of the Laughlin state, here at $(K_x = 0, K_y = 0)$ and $(K_x = 0, K_y = 2)$. The red line denotes the separation between the Chern insulator phase (upper part) and the trivial phase (lower part). *Right panel:* The corresponding spread δ (displayed as $1 - \min(\delta/\Delta, 1)$) as a function of ϕ and M/t_1. A color value of 1 would correspond to a perfectly twofold degenerate groundstate (i.e. $\delta = 0$).

B. The single-particle background

We now consider the one-body hamiltonian properties. In the left panel of Fig. 4, we show the non-universal nature of g - the averaged over BZ diagonal element of the quantum distance (FS) metric, more precisely $gA_{\rm BZ}$ (where $A_{\rm BZ}$ is the BZ area), is illustrated for the (two band) Haldane model with fixed parameters $t_1 = t_2 = 1$. The standard deviations divided by the averaged values g and the standard deviations of the Berry curvatures of the Haldane model are shown in the middle and right panel of Fig. 4. Note that both the relative deviations of the FS metric and those of the Berry curvature are minimal around the point $(M, \phi) = (0, 0.11\pi)$. This is around the same region that we have observed the strongest Laughlin-like state in our finite size numerical calculations.

To provide a more detailed insight of this point $(M, \phi) = (0, 0.11\pi)$, we provide in Fig. 5 the values of the g_{ij}^{FS} tensor (in an orthogonal coordinate system): $g_{11}^{FS}, g_{12}^{FS}, g_{21}^{FS}$, and g_{22}^{FS} as functions of the Bloch momentum in the Brillouin zone. We also give in Fig. 6 the values of the Berry curvature at the same point in the phase space.

C. Discussion

In this example of bosonic FCI states based on the Haldane model we found that, after the usual but simplifying procedure of the band flattening the Laughlin-

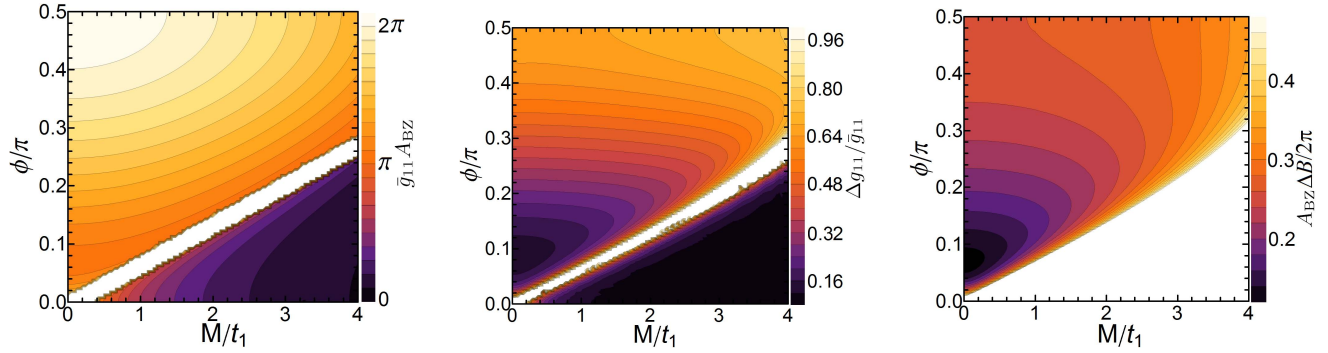

FIG. 4: *Left panel:* $\bar{g}_{11}A_{\rm BZ} = gA_{\rm BZ}$ for the Haldane model as a function of ϕ and M with $t_1 = t_2 = 1$. *Middle panel:* The corresponding deviation of g_{11}, i.e. $\Delta g_{11}/\bar{g}_{11}$. *Right panel:* The relative deviation of the Berry curvature B with respect to its averaged value.

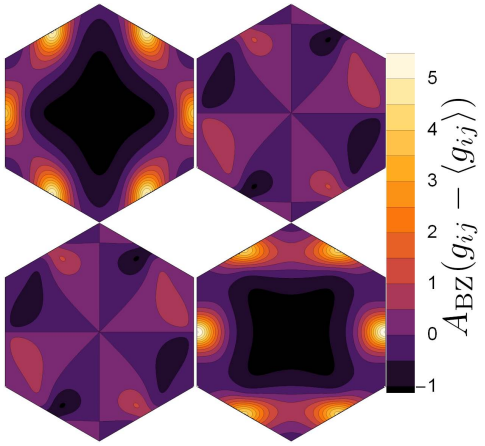

FIG. 5: The values of the FS metric with respect to its mean values at $(M, \phi) = (0, 0.11\pi)$. The mean values are $A_{\rm BZ}\langle g_{11}\rangle = A_{\rm BZ}\langle g_{22}\rangle = 1.13$ and $\langle g_{12}\rangle = \langle g_{21}\rangle = 0$ in the units of $A_{\rm BZ}|\overline{B}| = 2\pi$. In the Figure the graphs are ordered as the metric matrix elements. All four metric matrix elements are zero at $\mathbf{k} = (0, 0)$ momentum.

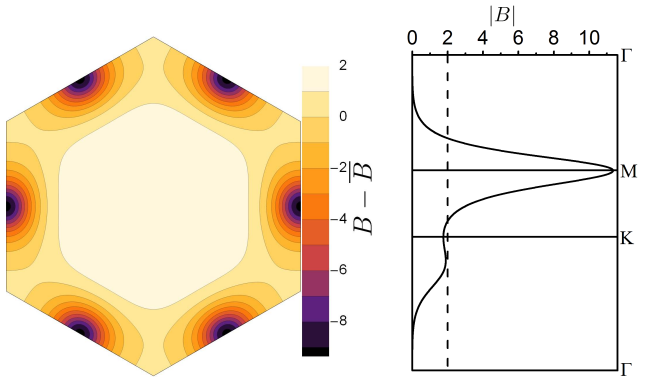

FIG. 6: The values of the Berry curvature in the BZ at $(M, \phi) = (0, 0.11\pi)$ with respect to its mean value, $\overline{B} = -2\pi/A_{\rm BZ}$, denoted by dash line. Plotted values of B are in the units of $\pi/A_{\rm BZ}$.

type $\nu = 1/2$ FCI state with the most significant gap and clear two-fold topological degeneracy corresponds to the phase-space parameters with the least dispersion of the Berry curvature and FS metric in the Brillouin zone, near FQHE values. Second, along $M = 0$ line, where we do not expect a formation of charge density wave but possibly a transition into another liquid state-superfluid, we observe the FCI state that is slowly modified but persistent with increase in ϕ angle, the value g of the FS metric and Berry curvature dispersion. The liquid transition should be accompanied with the collapse of a magneto-phonon gap and non-analytical behavior in the SSF. If we approximate the last term in Eq. 40 with $n|q|^4 g^2/2$ and then examine the expression for the projected SSF in the same equation for the values of g given in the left panel of Fig. 4, we see that we are approaching the transition around $\phi = \pi/2$ according to the mean field treatment.

VI. CONCLUSIONS

Based on the SSF calculations we studied the role of band geometry in the context of the FCI physics. We identified that the averaged over BZ FS metric plays the role of the quantum distance metric in the long-wave length domain based on the calculations of the single particle part of the projected to the band SSF. We discussed the behavior of the complete projected SSF in a mean-field framework, and whether and under which conditions the scenario of Ref. [22] is possible in the FCI context. We illustrated the role of the band geometry in the phase diagram of interacting bosons that live on the lattice of the Haldane model. The Laughlin $\nu = 1/2$ bosonic FCI state is the most pronounced for the FQHE value of the metric, i.e. when the fluctuations of the FS metric and Berry curvature are minimal. The Laughlin-like phase persists with the metric increase for a while and approaches a transition in an agreement with the mean field treatment of the long-distance physics of FCI. Further investigations are necessary which may provide us also with reasons for occurrence and stability of FCI

states.

VII. ACKNOWLEDGMENT

We thank A. Bernevig, M. Goerbig, and F.D.M. Haldane for discussions. This work was supported by the Serbian Ministry of Education and Science under projects No. ON171017 and ON171027. The authors also acknowledge support form the bilateral MES-CNRS 2011/12 program. N.R. was supported by NSF CAREER DMR-095242, ONR-N00014-11-1-0635, ARMY-245-6778, MURI-130-6082, Packard Foundation, and Keck grant. M.V.M. was supported by ONR-N00014-11-1-0635.

Appendix A: FCI state-single particle contribution

Here we analyze the expansion in small momentum to the quartic order of the expression in Eq. 24 that represents the single particle contribution to the projected SSF. To fourth order with assumed summations on repeated indices i, j, k

$$u_{\alpha,p-q} = u_{\alpha,p} - q_i\partial_i u_{\alpha,p} + \frac{q_iq_j}{2}\partial_i\partial_j u_{\alpha,p} - \frac{q_iq_jq_k}{3!}\partial_i\partial_j\partial_k u_{\alpha,p} + o(q^4), \tag{A1}$$

and therefore

$$\begin{aligned} u_{\alpha_1,p-q}u^*_{\alpha_2,p-q} &= u_{\alpha_1,p}u^*_{\alpha_2,p} - q_i u_{\alpha_1,p}\partial_i u^*_{\alpha_2,p} + \\ &\frac{q_iq_j}{2}u_{\alpha_1,p}\partial_i\partial_j u^*_{\alpha_2,p} - \frac{q_iq_jq_k}{3!}u_{\alpha_1,p}\partial_i\partial_j\partial_k u^*_{\alpha_2,p} \\ &-q_i\partial_i u_{\alpha_1,p}\ u^*_{\alpha_2,p} + q_iq_j\partial_i u_{\alpha_1,p}\partial_j u^*_{\alpha_2,p} \\ &-q_i\frac{q_kq_j}{2}\partial_i u_{\alpha_1,p}\partial_i\partial_j u^*_{\alpha_2,p} \\ &+\frac{q_iq_j}{2}\partial_i\partial_j u_{\alpha_1,p}\ u^*_{\alpha_2,p} - \frac{q_iq_j}{2}q_k\partial_i\partial_j u_{\alpha_1,p}\ \partial_k u^*_{\alpha_2,p} \\ &-\frac{q_iq_jq_k}{3!}\partial_i\partial_j\partial_k u_{\alpha_1,p}\ u^*_{\alpha_2,p} + o(q^4) \end{aligned} \tag{A2}$$

Because $u_{\alpha,p}u^*_{\alpha,p} = 1$ and therefore $u_{\alpha,p}\partial_i u^*_{\alpha,p} + \partial_i u_{\alpha,p}\ u^*_{\alpha,p} = 0$ we have to second order

$$\langle\tilde{\rho}_{-q}\tilde{\rho}_q\rangle|_{\text{single}} = n + o(q^2). \tag{A3}$$

To the second order we have

$$\begin{aligned} u^*_{\alpha_1,p}u_{\alpha_1,p-q}u^*_{\alpha_2,p-q}u_{\alpha_2,p} &\approx 1 + \\ \frac{q_iq_j}{2}&[u_{\alpha,p}\partial_i\partial_j u^*_{\alpha,p} + \partial_i\partial_j u_{\alpha,p}\ u^*_{\alpha,p} \\ &+2\ \partial_i u_{\alpha_1,p}\ u^*_{\alpha_1,p}\ u_{\alpha_2,p}\partial_i u^*_{\alpha_2,p}] \end{aligned} \tag{A4}$$

Due to

$$\begin{aligned} A_{ij}&[u_{\alpha,p}\partial_i\partial_j u^*_{\alpha,p} + \partial_i\partial_j u_{\alpha,p}\ u^*_{\alpha,p} \\ &+2\partial_i u_{\alpha_1,p}\partial_j u^*_{\alpha_1,p}] = 0 \end{aligned} \tag{A5}$$

for any symmetric A_{ij} we have

$$\langle\tilde{\rho}_{-q}\tilde{\rho}_q\rangle|_{\text{single}} = n - q_iq_j\sum_p g^{FS}_{ij}(p)n_p + o(q^3) \tag{A6}$$

where

$$\begin{aligned} g^{FS}_{ij}(p) = \frac{1}{2}&[\partial_i u_{\alpha_1,p}\partial_j u^*_{\alpha_1,p} + \partial_j u_{\alpha_1,p}\partial_i u^*_{\alpha_1,p} - \\ &\partial_i u_{\alpha_1,p}u^*_{\alpha_1,p}u_{\alpha_2,p}\partial_j u^*_{\alpha_2,p} - \partial_j u_{\alpha_1,p}u^*_{\alpha_1,p}u_{\alpha_2,p}\partial_i u^*_{\alpha_2,p}] \end{aligned} \tag{A7}$$

is the Fubini-Study metric. To the third order we find

$$\begin{aligned} u^*_{\alpha_1,p}u_{\alpha_1,p-q}u^*_{\alpha_2,p-q}u_{\alpha_2,p} &\approx 1 + q_iq_jg^{FS}_{ij}(p) \\ -\frac{q_iq_jq_k}{3!}&[u_{\alpha,p}\partial_k\partial_i\partial_j u^*_{\alpha,p} + \partial_k\partial_i\partial_j u_{\alpha,p}\ u^*_{\alpha,p} \\ &+3\partial_i u_{\alpha_1,p}u^*_{\alpha_1,p}u_{\alpha_2,p}\partial_k\partial_j u^*_{\alpha_2,p} + \\ &3\partial_i\partial_k u_{\alpha_1,p}u^*_{\alpha_1,p}u_{\alpha_2,p}\partial_j u^*_{\alpha_2,p}] + o(q^4) \end{aligned} \tag{A8}$$

Again differentiating $u_{\alpha,p}u^*_{\alpha,p} = 1$ three times we have

$$\begin{aligned} A_{ijk}&[u_{\alpha,p}\partial_i\partial_j\partial_k u^*_{\alpha,p} + \partial_i\partial_j\partial_k u_{\alpha,p}\ u^*_{\alpha,p} \\ &+3\partial_i\partial_k u_{\alpha_1,p}\partial_j u^*_{\alpha_1,p} \\ &+3\partial_i u_{\alpha_1,p}\partial_j\partial_k u^*_{\alpha_1,p}] = 0 \end{aligned} \tag{A9}$$

and therefore

$$\begin{aligned} u^*_{\alpha_1,p}u_{\alpha_1,p-q}u^*_{\alpha_2,p-q}u_{\alpha_2,p} &\approx 1 + q_iq_jg^{FS}_{ij}(p) + \\ \frac{q_iq_jq_k}{2}&[\partial_k u_{\alpha,p}\partial_i\partial_j u^*_{\alpha,p} + \partial_i\partial_j u_{\alpha,p}\partial_k u^*_{\alpha,p} \\ &-\partial_i u_{\alpha_1,p}u^*_{\alpha_1,p}u_{\alpha_2,p}\partial_k\partial_j u^*_{\alpha_2,p} + \\ &-\partial_i\partial_k u_{\alpha_1,p}u^*_{\alpha_1,p}u_{\alpha_2,p}\partial_j u^*_{\alpha_2,p}] = \\ 1 + q_iq_jg^{FS}_{ij}(p) &+ \frac{q_iq_jq_k}{2}\partial_k g^{FS}_{ij}(p) \end{aligned} \tag{A10}$$

Appendix B: SSF definition

By expanding the expression in Eq. 45 we immediately see that the quadratic contribution in q will be,

$$\sum_{i,j}|q|^2(R^\dagger_iR_j + R_iR^\dagger_j), \tag{B1}$$

and it is true that even when the condition in Eq. 46 is applied we have a non-zero contribution to the quadratic order. The expression in Eq. 45 does not correspond to the usual definition of the SSF as in Ref. 22, and only after an additional subtraction it reproduces the well-known behavior in the classical (continuum) FQHE.

We will illustrate and explain the source of this discrepancy in the coordinate representation of the continuum FQHE. First, by using the Eqs. 10-14 we see that [22]

$$\tilde{s}(q) = s(q) - n(1 - \exp\{-\frac{|q|^2}{2}\}). \tag{B2}$$

(This is most easily seen in the first quantization picture of the many-body problem considering the action of the

translation operator in the LLL, $\exp\{iq\partial/\partial z\}$; we rederived Eq. B2 in Section II A in the second quantization as a step towards the discussion of the FCI state.)

To extract the quadratic contribution to $\tilde{s}(q)$ we consider the expansion of the unprojected SSF, $s(q)$, and the following correlator of unprojected densities,

$$\frac{1}{V}\sum_{i,j}\langle \exp\{i\mathbf{q}\cdot\mathbf{r}_i\}\exp\{-i\mathbf{q}\cdot\mathbf{r}_j\}\rangle. \tag{B3}$$

If we assume the conservation of the angular momentum and use the complex notation of the LLL for the quadratic term, after the subtraction of "self-terms" i.e. these generated by $\sum_i\langle\exp\{i\mathbf{q}\cdot\mathbf{r}_i\}\rangle$, we have for the quadratic term the following expressions,

$$\begin{aligned}
&\frac{1}{V}\sum_{i,j}\langle \mathbf{q}\cdot\mathbf{r}_i\mathbf{q}\cdot\mathbf{r}_j\rangle = \\
&\frac{1}{V}\frac{|q|^2}{4}\sum_{i,j}\langle z_i^* z_j + z_j^* z_i\rangle = \\
&\frac{1}{V}\frac{|q|^2}{2}\sum_{i,j}\langle \frac{\partial}{\partial z_i}z_j + \frac{\partial}{\partial z_j}z_i\rangle = \\
&n|q|^2
\end{aligned} \tag{B4}$$

To get the final expression we used the properties of the LLL functions (which are holomorphic up to the Gaussian factor) and the property of the homogeneity of the ground state i.e. that its holomorphic part is annihilated by the $\sum_i \partial/\partial z_i$ operator (Eq. 46) in the momentum representation).

The substitution of the expression (B4) for $s(q)$ in Eq. B2 would lead to a non-zero, quadratic in q contribution to $\tilde{s}(q)$. This difference in the value that we have for $s(q)$ must stem from a difference in the subtractions: the one used in Ref. 22 and the other, when in a static correlator the ground state values of the two densities are subtracted ("self-terms"), implied by the expression of Eq. 45. Namely, in the classical reference the subtraction (i.e. a procedure to avoid divergences) is introduced for $s(q)$ at any $q \neq 0$ as,

$$s(q) = n + n^2\int d\mathbf{r}(g(r)-1)\exp\{-i\mathbf{q}\cdot\mathbf{r}\}, \tag{B5}$$

because the combination, $(g(r)-1)$, leads to the absence of the divergences for large r. For example, in the integer QH case we have (as an exact expression) $g(r) = 1-\exp\{-|q|^2/2\}$, and this leads to the usual, well-known behavior $s(q)\approx n|q|^2/2$ and $\tilde{s}(q)=0$. In Ref. 22 it was shown that $s(q)\approx n|q|^2/2$ for any liquid ground state of the system that conserves the angular momentum and particle number. The difference between this conclusion and the result in Eq. B4 stems from different subtraction procedures, and can be traced back to two different definitions of the SSF, the first in Eq. B5 that is related to a static limit of the time ordered density-density correlator, and the second, defined as a static correlator from which "self-terms" are subtracted (as in the expression of Eq. 45 in the momentum representation in the projected case).

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